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1991 J. Phys. A: Math. Gen. 24 1199

(http://iopscience.iop.org/0305-4470/24/6/013)

J. Phys. A: Math. Gen. 24 (1991) 1199-1214. Printed in the UK

Quantization of constrained systems and path integrals in curvilinear supercoordinates

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Received 6 February 1990

Abstract. For systems with constraints the issue of non-commutability of quantization and elimination of non-physical variables is studied in the framework of path integrals (PIs). It is shown that one should take into consideration the curvilinearity of physical variables and their phase space reduction in order to provide a one-to-one correspondence between the Dirac scheme and PI description. The latter leads to a modification of the standard PI (PI with a gauge condition). A general PI derivation is suggested for any method of picking out physical variables which corresponds to the Dirac scheme.

1. Introduction

It is well known that elimination of non-physical variables in gauge theories and quantization do not commute (Christ and Lee 1980, Prokhorov 1982b, Ashtekar and Horowitz 1982, Isham 1986). In other words, a quantum theory described by the Dirac scheme (Dirac 1965) can differ from that in which non-physical degrees of freedom are eliminated before quantization. However, the standard method of path integral (PI) construction (Faddeev 1970, Faddeev and Slavnov 1980) corresponds only to the last method since, in this way, non-physical momenta and coordinates are eliminated from the classical action with the help of constraints and supplementary conditions, respectively, and the phase space of physical degrees of freedom is *a priori* assumed to be an even-dimensional Euclidean space.

The difference between these quantization methods comes from the curvilinearity of physical variables (Prokhorov 1982a, b) (it is known that the application of operations of quantization and introduction of curvilinear coordinates in a different order to a classical theory give different quantum theories), on the one hand, and from their phase space reduction appearing because of a gauge symmetry (Prokhorov 1982b, Prokhorov and Shabanov 1989), on the other hand. A modification of PIs when a physical phase space is reduced was shown in Prokhorov and Shabanov (1989) and Shabanov (1989a, b, c, 1991). Other examples of the 'quantum-dynamical' phase space reduction were given by Dunne *et al* (1989).

The present work is devoted to the consideration of a PI form corresponding uniquely to the Dirac quantization scheme. It turns out that a connection exists between a PI form in curvilinear coordinates (section 2.3) and PIs for gauge theories containing both boson and fermion degrees of freedom. Existence of fermions in a theory causes, in PI derivation on physical superspace, some specific difficulties since one cannot

0305-4470/91/061199+16$03.50

decrease the number of anticommuting variables describing fermions by gauge transformations (section 4). In section 5 it is shown that taking into consideration the curvilinearity of physical variables and their phase space reduction, we may explicitly define the gauge-invariant kernel of the evolution operator via PIs. A mathematical reason for this is also presented in this section. In conclusion we suggest a general method for PI construction corresponding to the Dirac quantization scheme for an arbitrary choice of physical variables. It should be remembered that invariant and non-invariant means of choosing physical degrees of freedom exist. The first corresponds to the introduction of gauge-invariant variables. However, a complete set of gauge invariants is not always known, so we are forced to use the second method when physical variables are separated by supplementary conditions from initial variables, i.e. by gauge fixing. In the method suggested below, we show how one should take into account the curvilinearity of physical variables and their phase space reduction in a non-invariant way of separating them for the PI deviation.

2. PIs in curvilinear coordinates on superspace

Consider a quantum mechanical system containing boson degrees of freedom as well as Grassman ones. We take the Hamiltonian as follows:

$$H = \tfrac{1}{2}p_a^2 + V(x, \psi^+, \psi) \tag{2.1}$$

where $[x_a, p_b] = i\delta_{ab}$ $(a, b = 1, 2, \ldots, M)$ and $[\psi_\alpha^+, \psi_\beta]_+ = \delta_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, N)$. The operator algebra may be realized in a space of functions on superspace $\Phi = \Phi(x, \bar\psi) \equiv \Phi(Q)$, $Q = (x, \bar\psi)$ ($\bar\psi$ is complex conjugated to ψ) if

$$p_a\Phi = -i\frac{\partial}{\partial x_a}\Phi \qquad \psi_\alpha^+\Phi = \bar\psi_\alpha\Phi \qquad \psi_\alpha\Phi = \frac{\partial}{\partial\bar\psi_\alpha}\Phi. \tag{2.2}$$

Here and below all derivations of Grassman variables are left. The scalar product under which we define Hermitian conjugated operators has the form (Berezin 1966)

$$\langle\Phi_1|\Phi_2\rangle = \int dx\, d\bar\psi\, d\psi\, e^{-\bar\psi\psi}\bar\Phi_1(Q)\Phi_2(Q) \tag{2.3a}$$

where the integral is taken over the whole $\mathbb{R}^M$. In accordance with (2.3*a*) the unit operator kernel $\langle Q|Q'\rangle$ has the form

$$\sum_E \Phi_E(Q)\bar\Phi_E(Q') \equiv \delta(Q, \bar Q') = \delta(x - x')\, e^{\bar\psi\psi'} \tag{2.3b}$$

where Φ_E are eigenfunctions of Hamiltonian (2.1).

In the general case the change of variables is defined by a function on superspace $Q = Q(q)$, $q = (y, \bar\xi)$. However, we shall consider special forms of Q, which will be enough for the application of gauge theories. Introduce the new variables

$$x_a = x_a(y) \qquad \psi_\alpha = \Omega_{\alpha\beta}\xi_\beta \tag{2.4}$$

where $\Omega \in SU(N)$ and $\Omega = \Omega(y)$. Then $dQ = A\, dq$ and $\partial/\partial Q = A^{-1T}\partial/\partial q$. Here $A_j^i = \partial Q^i/\partial q^j$, $i, j = (a, \alpha)$. After some calculations we get from (2.4)

$$\frac{\partial}{\partial x_a} = B_a^b(\partial_b + i\pi_b) \tag{2.5}$$

where $B_a^b = [(\partial x/\partial y)^{-1}]_a^b$, $\pi_b = i\bar{\xi}\partial_b \Omega^+ \Omega \partial/\partial\bar{\xi}$, $\partial_b = \partial/\partial y_b$. Using (2.5), (2.4) and (2.2), we rewrite the Hamiltonian (2.1) in the new variables:

$$H = \tfrac{1}{2} P_a g^{ab} P_b + V_q(y) + V. \tag{2.6}$$

Here $P_a = -i\mu^{-1/2}(\partial_a + i\pi_a) \circ \mu^{1/2}$, $\mu = \operatorname{sdet} A = \sqrt{g}$, $g = \det\|g^{ab}\|^{-1}$, $g^{ab} = B_c^a B_c^b$ and V_q is the effective quantum correction to a potential:

$$V_q = \frac{1}{2\sqrt{\mu}} (\partial_a g^{ab}) \partial_b \sqrt{\mu} + \frac{1}{2\sqrt{\mu}} g^{ab} \partial_a \partial_b \sqrt{\mu}. \tag{2.7}$$

The form of a scalar product in the space of functions $\varphi = \varphi(q)$ follows from (2.3*a*) and (2.4)

$$\langle \varphi_1 | \varphi_2 \rangle = \int_K dy\, d\bar{\xi}\, d\xi\, e^{-\bar{\xi}\xi} \mu(y) \bar{\varphi}_1(q) \varphi_2(q) \tag{2.8}$$

where K is a region of integration over y, $K \subset \mathbb{R}^M$. The mapping $x = x(y)$ is one to one if $x \in \mathbb{R}^M$ and $y \in K$. Let the functions $x(y)$ be analytical for all $y \in \mathbb{R}^M$. So, there exist transformations $\hat{s}$ acting in $\mathbb{R}^M$ such as

$$x_a(y) = x_a(\hat{s}y). \tag{2.9}$$

For example, $M = 2$ and $y_a = (r, \theta)$ are polar coordinates. In this case K is the strip $r > 0$, $\theta \in [0, 2\pi)$, and transformations $\hat{s}$ have the form $\theta \to \theta + 2\pi n$, $n \in \mathbb{Z}$ and $r \to -r$, $\theta \to \theta + \pi$. Apparently, transformations $\hat{s}$ form a discrete group S and $K = \mathbb{R}^M/S$, i.e. K is made from $\mathbb{R}^M$ by identification of points in $\mathbb{R}^M$ connected with each other by S-transformations. In mathematical language, the mapping $x = x(y)$ gives the projection in the principal fibre bundle (Kobayashi and Nomizu 1963) where the base and fibre bundle coincide with $\mathbb{R}^M$ and the discrete group S acts along a fibre in the fibre bundle $\mathbb{R}^M(y)$.

The group S induces discrete transformations of Grassman variables $\hat{T}_s$ such as $\Omega(y)\xi = \Omega(\hat{s}y)\hat{T}_s\xi$, so, $\hat{T}_s = \Omega^+(\hat{s}y)\Omega(y)$. We mark the total group of transformations $\hat{s}$ and $\hat{T}_s$ as S^* and $\hat{s}^* q = (\hat{s}y, \bar{\xi}\hat{T}_s^+)$.

Hamiltonian (2.6) is written in an explicit Hermitian form since the operators P_a are Hermitian under scalar product (2.8). If $\varphi_{E'}$ are eigenfunctions of (2.6), we may write (since the Hilbert spaces (2.1) with (2.3*a*) and (2.6) with (2.8) are isomorphic)

$$\varphi_{E'}(q) = \sum_E c_{E'E} \Phi_E(Q). \tag{2.10}$$

By definition $Q(\hat{s}^* q) = Q(q)$, and we conclude that

$$\varphi_{E'}(\hat{s}^* q) = \varphi_{E'}(q). \tag{2.11}$$

Using the property of parity (2.11), we may analytically continue φ_E to the non-physical region $y \in \mathbb{R}^M$. Thus, in accordance with (2.8) we have

$$\sum_E \varphi_E(q) \bar{\varphi}_E(q') = \sum_{S^*} (\mu(y)\mu(\hat{s}y'))^{-1/2} \delta(q, \hat{s}^* \bar{q}') \tag{2.12a}$$

where $y \in \mathbb{R}^M$, $y' \in K$. For physical values of y, y', i.e. y, $y' \in K$, one should only keep the first term in sum (2.12*a*) with $\hat{s}^* = 1$. Formula (2.12*a*) defines an analytical continuation of the unit operator kernel $\langle q | q' \rangle$ to the non-physical region.

Note also that kernel (2.12*a*) can be obtained directly from (2.3*b*). Since a change of variables in a quantum theory is equivalent to a choice of a new basis in the Hilbert space of states, the left-hand sides of (2.3*b*) and (2.12*a*) must coincide, hence, the

right-hand sides coincide too. Indeed, let us change Q and Q' in (2.3b) by expressions (2.4) and assume $y \in \mathbb{R}^M$, $y' \in K$, then

$$\delta(Q, \bar{Q}') = \sum_S [\mu(y)\mu(\hat{s}y')]^{-1/2} \delta(y - \hat{s}y') \exp \bar{\xi} \hat{T}_s \xi'. \tag{2.12b}$$

The equality follows from the rule of changing an argument of M-dimensional δ-function and the definition of $\hat{T}_s = \Omega^+(\hat{s}y')\Omega(y')$ (we may change $\Omega^+(y)$ to $\Omega^+(\hat{s}y')$ with $\exp \bar{\psi}\psi'$ in front of $\delta(y - \hat{s}y')$).

Let us turn now directly to the PI derivation. The kernel of the infinitesimal evolution operator is

$$U_\varepsilon(q, \bar{q}') = [1 - i\varepsilon H(y, \bar{\xi}, \partial_{\bar{\xi}})]\langle q|q'\rangle \tag{2.13}$$

where H is given by (2.6) and $\varepsilon \to 0$. We rewrite kernel (2.12a) as follows:

$$\int \frac{dy''\, d\bar{\xi}''\, d\xi''\, e^{-\bar{\xi}''\xi''}}{(\mu\mu'')^{1/2}} \delta(q, \bar{q}'') Q(q'', \bar{q}') \tag{2.14}$$

where $\mu = \mu(y)$, $\mu'' = \mu(y'')$ and

$$Q(q'', \bar{q}') = \sum_{S^*} \delta(q'', \hat{s}^* \bar{q}'). \tag{2.15}$$

Then, we use the representation of δ-function $\delta(y) = (2\pi)^{-M} \int dp\, e^{ipy}$ in (2.14) and substitute (2.14) into (2.13). For the calculation of the action of H on $\delta(q, \bar{q}'')$ one should take into consideration a non-commutability $\bar{\xi}$ and $\partial_{\bar{\xi}}$ and also use the equality

$$\partial_a \circ g^{ab}(y)\partial_b \delta(y - y'') = (g^{ab}(y'')\partial_a\partial_b - \partial_a g^{ab}(y'')\partial_b)\delta(y - y'') \tag{2.16}$$

where $\partial_a = \partial/\partial y_a$. Thus we find, with an accuracy of $O(\varepsilon^2)$,

$$U_\varepsilon(q, \bar{q}') = \int \frac{dy''\, d\bar{\xi}\, d\xi''}{(\mu\mu'')^{1/2}} \exp(-\bar{\xi}''\xi'') U_\varepsilon^{\text{eff}}(q, \bar{q}'') Q(q'', \bar{q}') \tag{2.17}$$

$$U_\varepsilon^{\text{eff}}(q, \bar{q}'') = \int \frac{dp}{(2\pi)^M} \exp(\bar{\xi}\xi'') \exp i\varepsilon \left(\frac{p}{\varepsilon}(y - y'') - H^{\text{eff}}(p, q, \bar{q}'') \right) \tag{2.18}$$

and the effective Hamiltonian has the form

$$H^{\text{eff}}(p, q, \bar{q}'') = H_0(p, q, \bar{q}'') + \tilde{V}(q, \bar{q}'') + \tilde{V}_q(p, q, \bar{q}'') \tag{2.19}$$

$$H_0 = \tfrac{1}{2}(p_a + \pi_a) g^{ab}(y'')(p_b + \pi_b) \tag{2.20}$$

where Grassman variables $\xi'' = \bar{\xi}''$ stand instead of $\partial_{\bar{\xi}}$ in π_a; $\tilde{V}$ follows from V if we carry all operators $\partial_{\bar{\xi}}$ to the right and then change them by Grassman variables ξ''; and the magnitude

$$\tilde{V}_q = V_q(y'') + \frac{i}{2} \partial_a g^{ab}(y'')(p_b + \pi_b) - \tfrac{1}{2} g^{ab}(y'') \bar{\xi} \partial_a \Omega^+(y'') \partial_b \Omega(y'') \xi'' \tag{2.21}$$

is the quantum correction to the potential. It takes into account the non-commutability of operators in the kinetic energy operator. If we restore the dependence on $\hbar$, then $V_q \sim \hbar^2$ and other terms in (2.21) $\sim \hbar$. This shows their connection with the operator ordering (see the review by Prokhorov (1982a) and references therein). When ε tends to zero, we can replace $y - y''$ by $\dot{y}''\varepsilon$ with an accuracy of $O(\varepsilon^2)$.

To obtain the evolution operator kernel for a finite time, we must find the formula for iterations of infinitesimal kernels (2.17). By definition (2.8) we write

$$U_{2\varepsilon}(q, \bar{q}') = \int_K \mathrm{d}y'' \, \mathrm{d}\bar{\xi}'' \, \mathrm{d}\xi'' \, \mu'' \, \mathrm{e}^{-\bar{\xi}''\xi''} U_\varepsilon(q, \bar{q}'') U_\varepsilon(q'', \bar{q}'). \tag{2.22}$$

Transformations (2.22) are cumbersome enough. However, we may easily control them if we take into account that their main sense is to carry to the right the operator $\hat{Q}$ being between two $U_\varepsilon^{\mathrm{eff}}$ in (2.22) (see (2.17)). In this way we would like to represent the final formula as (2.17) where $\varepsilon \to 2\varepsilon$ and

$$U_{2\varepsilon}^{\mathrm{eff}}(q, \bar{q}') = \int \mathrm{d}y_1 \, \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\xi_1 \, \mathrm{e}^{-\bar{\xi}_1\xi_1} U_\varepsilon^{\mathrm{eff}}(q, \bar{q}_1) U_\varepsilon^{\mathrm{eff}}(q_1, \bar{q}'). \tag{2.23}$$

If into (2.22) we place expression (2.17) instead of the first U_ε, the integration is carried out over the right argument of the kernel Q and over the left argument of the second U_ε entering into (2.22). Let us calculate, at first, the action of $\hat{Q}$ from the left on the function Φ. We have in accordance with (2.15)

$$\hat{Q}\Phi(q) = \int_K \mathrm{d}y' \, \mathrm{d}\bar{\xi}' \, \mathrm{d}\xi' \, \mathrm{e}^{-\bar{\xi}'\xi'} Q(q, \bar{q}')\Phi(q') = \int_K \mathrm{d}y' \sum_S \delta(y - \hat{s}y')\Phi(y', \bar{\xi}\hat{T}_s). \tag{2.24}$$

To take the integral over y', we rearrange Σ_S and $\int_K$ and change integration variables $z = \hat{s}y'$. In the general case, $\hat{s}$ is not a linear transformation, i.e., $\hat{s}y = s(y)$ is a certain function. So,

$$\hat{Q}\Phi(q) = \sum_S \theta_K(\hat{s}^{-1}y) \frac{1}{J_s(y)} \Phi(\hat{s}^{-1}y, \bar{\xi}\hat{T}_s). \tag{2.25}$$

Here $J_s = D(\hat{s}y)/D(y)$ is Jacobian,

$$\theta_K(y) = \begin{cases} 1 & y \in K \\ 0 & y \bar{\in} K. \end{cases} \tag{2.26}$$

Apparently, the measure $\mathrm{d}x = \mathrm{d}x(y)$ is invariant under the group S hence from the equality $\mathrm{d}x(\hat{s}y) = \mathrm{d}x(y)$ it follows that

$$\mu(\hat{s}y) = (J_s(y))^{-1}\mu(y). \tag{2.27}$$

Using the property (2.27) and $\hat{T}_s = \hat{T}_{s^{-1}}^{+}$ with (2.25) we can take the integrals over y'', $\bar{\xi}''$ and ξ'' in (2.22):

$$U_{2\varepsilon}(q, \bar{q}') = \int \frac{\mathrm{d}y_1 \, \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\xi_1}{(\mu\mu_1)^{1/2}} \mathrm{e}^{-\bar{\xi}_1\xi_1} U_\varepsilon^{\mathrm{eff}}(q, \bar{q}_1) \sum_S \mu_1 \theta_K(\hat{s}y_1) U_\varepsilon(\hat{s}^* q_1, \bar{q}'). \tag{2.28}$$

By construction,

$$U_\varepsilon(\hat{s}^* q, \bar{q}') = U_\varepsilon(q, \bar{q}'). \tag{2.29}$$

Indeed, since the initial Hamiltonian (2.1) (or (2.6), which is the same) is invariant under S^* ($Q(\hat{s}^*q) = Q(q)$!), by definition (2.13) we conclude that (2.29) follows from the equality $\langle \hat{s}^* q | q' \rangle = \langle q | q' \rangle$, which must take place in accordance with definition (2.12*a*) and the parity (2.11). Of course, we may directly prove the symmetry property of the unit operator kernel under S^* making calculations like (2.24) and (2.25). The result of the action of kernel (2.12*a*) from the left on the function Φ coincides with the right-hand side of (2.25) if the factor $(J_s(y))^{-1}$ is omitted. Thus, the function

$\tilde{\Phi}(q)=\Sigma_{S^*}\theta_K(\hat{s}y)\Phi(\hat{s}^*q)$ is invariant under S^*. If Φ belongs to the Hilbert space of the theory, i.e. it is a linear combination of $\varphi_E(q)$, then $\Phi(\hat{s}^*q)=\Phi(q)$. So, the equality $\tilde{\Phi}=\Phi$ follows from

$$\sum_S \theta_K(\hat{s}y)=1. \tag{2.30}$$

Now we can see from (2.29) and (2.30) that the summation over S^* in (2.28) disappears. After substituting (2.17) into (2.28) we find the required expression for $U_{2\varepsilon}$ coinciding with (2.17) if $\varepsilon\to 2\varepsilon$, and $U_{2\varepsilon}^{\text{eff}}$ is defined by (2.23).

Now, clearly, all iterations of U_ε are reduced to iterations of $U_\varepsilon^{\text{eff}}$. On the other hand, iterations of the kernel $U_\varepsilon^{\text{eff}}$ give the standard finite-dimensional approximation of PIs (Feynman and Hibbs 1965) for the theory with Hamiltonian (2.19). Thus we get for a finite time interval

$$U_t(q,\bar{q}')=\int \frac{\mathrm{d}y''}{(\mu\mu'')^{1/2}}\,\mathrm{d}\bar{\xi}''\,\mathrm{d}\xi''\,\mathrm{e}^{-\bar{\xi}''\xi''}\,U_t^{\text{eff}}(q,\bar{q}'')Q(q'',\bar{q}') \tag{2.31}$$

where the kernel U_t^{eff} has the standard PI form

$$U_t^{\text{eff}}(q,\bar{q}'')=\int \prod_{\tau=0}^{t}\left(\frac{\mathrm{d}p\,\mathrm{d}y}{(2\pi)^M}\,\mathrm{d}\bar{\xi}\,\mathrm{d}\xi\right)\mathrm{e}^{\gamma}\,\mathrm{e}^{\mathrm{i}S_{\text{eff}}}. \tag{2.32}$$

Here $\gamma=\frac{1}{2}(\bar{\xi}(t)\xi(t)+\bar{\xi}(0)\xi(0))$ takes into account the standard initial conditions in PIs containing Grassman variables (Faddeev and Slavnov 1980) $\bar{\xi}(t)=\bar{\xi}$ and $\xi(0)=\xi''$. Moreover, $y(t)=y$ and $y(0)=y''$ are initial conditions for boson variables, and

$$S_{\text{eff}}=\int_0^t \mathrm{d}\tau\left(p\dot{y}+\frac{1}{2\mathrm{i}}(\bar{\xi}\dot{\xi}-\dot{\bar{\xi}}\xi)-H^{\text{eff}}\right). \tag{2.33}$$

Note that the measure μ (Jacobian) is not contained in the PI measure, but it stays as a factor both at initial and finite points of the transition amplitude. If we omit the dependence of the theory on Grassman degrees of freedom, the boson PI in curvilinear coordinates appears for which the recipe of construction was suggested by Prokhorov (1984) (see also his review (Prokhorov 1982a)).

The main difficulty appearing in the PI derivation in curvilinear coordinates is that a physical region of values for new variables is reduced $\dot{\mathbb{R}}^M\to K\subset\mathbb{R}^M$. Moreover eigenvalues of some canonical momenta become discrete (e.g. the angular momentum, see section 3), i.e. integration over them is replaced by summation.

We have got over these difficulties by using the analytical continuation of the unit operator kernel (2.12*a*) in the PI derivation. We have found that the integration in PIs can be carried out over the total phase space $\mathbb{R}^M\otimes\mathbb{R}^M$; however, after calculation of a transition amplitude we must symmetrize it with respect to the group S^* in accordance with (2.31).

3. Example: two-dimensional SUSY oscillator

In this short section we give a simple illustration of general formulae of section 2. Consider a two-dimensional SUSY oscillator. Its Hamiltonian is

$$H=-\tfrac{1}{2}\Delta+\tfrac{1}{2}x_a^2+\psi_a^+\psi_a-1 \qquad a=1,2. \tag{3.1}$$

Let us study states of this oscillator with a fixed total angular momentum, i.e. with a total angular momentum of bosons and fermions. For this we introduce the generalized polar coordinates

$$x_1 = r\cos\theta \qquad x_2 = r\sin\theta \qquad \psi_a = e^{i\theta}\xi_a. \tag{3.2}$$

In this case $\mu = r$, $g^{ab} = \mathrm{diag}(1, r^{-2})$, $P_1 = p_r = -ir^{-1/2}\partial_r \circ r^{1/2}$ is the Hermitian momentum operator for a radial degree of freedom, $P_2 = P_\theta = -i\partial_\theta + \xi_a^+\xi_a$ is the angular momentum of a boson (a total angular momentum is $-i\partial_\theta$) and $V_q = -\frac{1}{8}r^{-2}$. The Hamiltonian in coordinates (3.2) is defined by (2.6).

To get the PI, one should find a group S^*. The structure of the group S for polar coordinates was discussed in section 2 where it was found that $K = \mathbb{R}^2 \backslash S$ is the strip $r > 0$, $\theta \in [0, 2\pi)$. So, $\hat{T}_s$ should only be calculated. Apparently, for $\hat{s}\theta = \theta + 2\pi n$, $\hat{s}r = r$ we have $\hat{T}_s = 1$ and for $\hat{s}\theta = \theta + \pi$, $\hat{s}r = -r$ the equality $\hat{T}_s\xi_a = -\xi_a$ takes place. Thus, the operator $\hat{Q}$ has the form

$$Q(q, \tilde{q}') = Q_1(\theta, \theta')\delta(r, r')\exp\bar{\xi}_a\xi'_a + Q_1(\theta, \theta' + \pi)\delta(r + r')\exp(-\bar{\xi}_a\xi'_a) \tag{3.3}$$

where

$$Q_1(\theta, \theta') = \sum_{n=-\infty}^{\infty} \delta(\theta - \theta' + 2\pi n). \tag{3.4}$$

The PI is obtained from (2.31).

4. A simple model with a gauge symmetry

Here we show the connection between the description of dynamical systems in curvilinear coordinates and gauge-invariant description of gauge models. Consider a mechanical model with the SO(3) gauge group and with the Lagrangian

$$L = \tfrac{1}{2}(D_t x)^2 + i\psi^+ D_t\psi - V(x, \psi^+, \psi) \tag{4.1}$$

where $x \in \mathbb{R}^3$, ψ is a three-dimensional complex Grassman vector, $D_t = \partial_t + y$, y is a real 3×3 antisymmetric matrix. Lagrangian (4.1) is invariant under gauge transformations

$$x \to \Omega x \qquad \psi \to \Omega\psi \qquad y \to \Omega y\Omega^T + \Omega\partial_t\Omega^T. \tag{4.2}$$

Here $\Omega = \Omega(t) \in \mathrm{SO}(3)$, and we assume also that V is gauge invariant.

Let us turn to the Hamiltonian formalism. Canonical momenta are $\pi = \partial L/\partial\dot{y} = 0$, $p = \partial L/\partial\dot{x} = D_t x$ and

$$\pi_{\psi^+} = \frac{\partial L}{\partial\dot{\psi}^+} = 0 \qquad \pi_\psi = \frac{\partial L}{\partial\dot{\psi}} = -i\psi^+. \tag{4.3}$$

Obviously, $\pi = 0$ and (4.3) gives primary constraints. Note, (4.3) are the second-class constraints (Dirac 1965) which appear always since usual Lagrangians for fermions are linear in velocities. To eliminate the second-class constraints, we replace the Poisson brackets (a definition of the Poisson brackets for Grassman variables was given by Martin (1959)) by Dirac brackets (Dirac 1965). We take ψ^+ and ψ as new canonical conjugate variables (Martin 1959), and their Dirac brackets are

$$\{\psi_a^+, \psi_b\}_D = \{\psi_b, \psi_a^+\}_D = -i\delta_{ab} \tag{4.4}$$

$a, b = 1, 2, 3$. The momenta π_ψ and π_{ψ^+} are eliminated from the theory by using constraints (4.3). The Hamiltonian of the system has the form

$$H = \tfrac{1}{2}p_a^2 + V(x, \psi^+, \psi) - pyx - \mathrm{i}\psi^+ y\psi. \tag{4.5}$$

Put $y_{ab} = y_c\varepsilon_{abc}$, where ε_{abc} is a completely antisymmetric tensor, $\varepsilon_{123} = 1$, and also $\pi_{ab} = \pi_c\varepsilon_{cab}$. So, the secondary constraints (Dirac 1965) are

$$\{\pi_a, H\} = \varepsilon_{abc}(p_b x_c + \mathrm{i}\psi_b^+\psi_c) \equiv G_a = 0. \tag{4.6}$$

Constraints (4.6) are the first-class constraints $\{G_a, G_b\}_{\mathrm{D}} = -\varepsilon_{abc}G_c$, $\{G_a, H\}_{\mathrm{D}} = \varepsilon_{abc}y_b G_c$ (Dirac 1965).

After quantization, when all canonical variables are replaced by operators and $\{,\}_{\mathrm{D}} \to -\mathrm{i}[,]$ ($[,]$ is a commutator for bosons and an anticommutator for fermions), constraints (4.6) pick out the physical subspace of states:

$$G_a|\Phi_{\mathrm{ph}}\rangle = 0 \qquad |\Phi_{\mathrm{ph}}\rangle \in \mathscr{H}_{\mathrm{ph}}. \tag{4.7}$$

(The constraints $\pi_a|\Phi_{\mathrm{ph}}\rangle = 0$ are easily solved, and we shall not turn our attention to them below.) To construct $\mathscr{H}_{\mathrm{ph}}$, we use the representation of second quantization $a_b = 1/\sqrt{2}(x_b + \mathrm{i}p_b)$. By definition, the vacuum is $a_b|0\rangle = \psi_b|0\rangle \equiv 0$. We find $\varepsilon_{abc}p_b x_c = -\mathrm{i}\varepsilon_{abc}a_b^+ a_c$ and $|0\rangle \in \mathscr{H}_{\mathrm{ph}}$. The operators of the constraints generate SO(3) rotations of the vectors a^+ and ψ^+, hence, $|\Phi_{\mathrm{ph}}\rangle$ may be obtained by action on $|0\rangle$ of all combinations of a^+ and ψ^+ which are invariant under SO(3) transformations. The basis for these polynomials of a^+ and ψ^+ is defined by invariant tensors of the group SO(3), δ_{ab} and ε_{abc} (Barut and Raczka 1977), i.e. we may obtain any $|\Phi_{\mathrm{ph}}\rangle$ acting on $|0\rangle$ by the operators

$$b_1^+ = a_c^+ a_c^+ \qquad b_2^+ = \varepsilon_{abc}\psi_a^+\psi_b^+ a_c^+ \tag{4.8a}$$

$$f_1^+ = a_b^+\psi_b^+ \qquad f_2^+ = \varepsilon_{abc}\psi_a^+\psi_b^+\psi_c^+. \tag{4.8b}$$

The operators (4.8*a*) are 'boson', i.e. they commute, and the operators (4.8*b*) correspond to physical excitations of a fermion sector, i.e. they anticommute.

Now we return to the PI derivation. Christ and Lee (1980) and Prokhorov (1982b) have shown for the model (4.1), but without fermions, that the elimination of non-physical variables and subsequent quantization lead to the results contradicting the Dirac scheme. The main point is as follows. Put, for example, $V = \frac{1}{2}x^2$ (fermions are absent), then the basis in $\mathscr{H}_{\mathrm{ph}}$ is $b_1^{+n}|0\rangle$, $n = 0, 1, \ldots$ (Prokhorov and Shabanov 1989), i.e. the oscillator spectrum is $E_n = 2n + \frac{3}{2}$. Now we eliminate non-physical variables before quantization. Since the constraints $G_a = l_a = \varepsilon_{abc}p_b x_c = 0$ are projections of the angular momentum of a boson, we conclude that angles of the spherical coordinate system $x \to (r, \theta, \varphi)$ are non-physical variables (their canonical momenta are $p_\varphi = -l_3 = 0$, $p_\theta = \sin\varphi\, l_1 - \cos\varphi\, l_2 = 0$). So, the classical physical Hamiltonian depending on physical variables r and $p_r = p_a x_a / r$ is $\frac{1}{2}(p_r^2 + r^2)$. It coincides with the Hamiltonian of a one-dimensional oscillator, the quantization of which ($p_r \to \mathrm{i}\partial_r$) gives the spectrum $E_n = n + \frac{1}{2}$.

On the other hand, as has been noted in section 1, a standard method of PI construction corresponds to a quantum theory obtained from an initial classical one just by eliminating non-physical degrees of freedom before quantization. From this point of view it is interesting to find a PI form which corresponds to the Dirac quantization scheme. With this purpose we, at first, quantize the theory, then eliminate non-physical variables and construct the quantum Hamiltonian in $\mathscr{H}_{\mathrm{ph}}$. Finally, using it we derive a PI for the evolution operator kernel in $\mathscr{H}_{\mathrm{ph}}$.

Note, to eliminate non-physical variables in a quantum theory, one should introduce curvilinear coordinates. Indeed, if we define new variables so that some of them get shifts (non-physical variables) and others do not change under gauge transformations (physical variables) constraints become diagonal, i.e. they are linear combinations of momentum operators conjugated to non-physical variables. It then follows that constraints are generators of gauge transformations, and momentum operators are generators of translations. However, if gauge transformations are isotopic rotations, we cannot diagonalize constraints without introducing curvilinear coordinates.

On this basis, we define new variables

$$\begin{matrix} x = U_\rho \\ \psi = U\xi \end{matrix} \qquad U = \begin{pmatrix} \sin\theta\cos\varphi & -\sin\varphi & -\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \cos\varphi & \cos\theta\sin\varphi \\ \cos\theta & 0 & \sin\theta \end{pmatrix} \in \mathrm{SO}(3) \tag{4.9}$$

where $\rho = (r, 0, 0)$ and $\xi = (\xi_a)$ $(a = 1, 2, 3)$. Apparently, θ and φ get shifts under gauge transformations and r, ξ_1 do not change (as was shown below $\xi_{2,3}$ are not guage-invariant). The quantum Hamiltonian in the new variables has the form (2.6) if $y_a = (r, \theta, \varphi)$, $\pi_r = 0$, $\pi_\theta = L_2$, $\pi_\varphi = \sin\theta\, L_3 + \cos\theta\, L_1$, $L_a = \mathrm{i}\varepsilon_{abc}\xi_b^+\xi_c$ $(\xi_c = \partial/\partial\bar{\xi}_c)$, $V_q = 0$, $\mu = r^2$ and the metric tensor is $g^{ab} = \mathrm{diag}(1, r^{-2}, (r\sin\theta)^{-2})$.

Straightforward calculations with the use of (2.5) show that the system of equations (4.7) in the coordinate representation is equivalent to

$$\partial_\varphi \Phi_{\mathrm{ph}} = \partial_\theta \Phi_{\mathrm{ph}} = 0 \qquad L_1 \Phi_{\mathrm{ph}} = 0. \tag{4.10}$$

The form of the third equation in (4.10) may easily be understood if we note that using gauge transformations we may always reduce φ, θ to zeros in $x = U\rho$; however, the vector ρ has the stationary subgroup SO(2) with the generator ε_{1ab} being a subgroup of the gauge group SO(3). These remaining gauge transformations do not change ρ, but they change $\xi_{2,3}$; hence, physical fermion states should be invariant under it, i.e. $L_1\Phi_{\mathrm{ph}} = 0$. So, the Hamiltonian in $\mathscr{H}_{\mathrm{ph}}$ is

$$H_{\mathrm{ph}} = -\frac{1}{2r}\partial_r^2 \circ r + \frac{1}{2r^2}(L_2^2 + L_3^2) + V(\rho, \xi^+, \xi). \tag{4.11}$$

A gauge symmetry in a pure fermion system was studied in Shabanov (1989a). Using gauge transformations we cannot decrease the number of Grassman variables. Nevertheless, in a classical theory the constraint of the type $L_1 = 0$ leads to that the time evolution of one fermion degree of freedom, for example, $\xi_2(t)$ is determined by the time evolution of the other, i.e. $\xi_3(t)$. In a quantum theory the constraint $L_1\Phi_{\mathrm{ph}} = 0$ is equivalent to the requirement of $\mathbb{Z}_2$-invariance: $\xi_{2,3} \to -\xi_{2,3}$ for Φ_{ph} (the latter was interpreted as a phase space reduction for a fermi-system (Shabanov 1989a). Thus, we find

$$\Phi_{\mathrm{ph}}(r, \hat{T}_1\bar{\xi}) = \Phi_{\mathrm{ph}}(r, \bar{\xi}) \tag{4.12}$$

where $T_1 = \hat{s}_1 = \mathrm{diag}(1, -1, -1) \in \mathrm{SO}(3)$. Note that the only invariant of the SO(2) subgroup generating L_1 and of $\mathbb{Z}_2 = (1, \hat{s}_1)$ is $\bar{\xi}_2\bar{\xi}_3$.

Let us define now the scalar product in $\mathscr{H}_{\mathrm{ph}}$. Since (4.9) is the change of variables, we make it like (2.8). However, as θ and φ are now non-physical variables, K is here a physical configuration space of r. The symmetry group of the change of variables (4.9) contains the group $S = \mathbb{Z}_2 : \rho \to \pm\rho$ acting on physical variables. So, $K = \mathbb{R}\backslash S$ is a semiaxis $r > 0$.

To find the full group S^* acting in a physical superspace, one should determine all non-trivial transformations $\hat{s} \in \mathrm{SO}(3)$ such as $\hat{s}\rho = \pm\rho$. They are $\hat{s}_1 = \mathrm{diag}(1, -1, -1)$, $\hat{s}_2 = \mathrm{diag}(-1, 1, -1)$ and $\hat{s}_3 = \mathrm{diag}(-1, -1, 1)$. Since $\hat{s}_a^2 = 1$ (no summation over a), we see that the following transformations of θ, φ correspond to $\hat{s}_a (x = U\rho = U\hat{s}_a\hat{s}_a\rho)$: $\hat{s}_1 : \theta \to -\theta, \varphi \to \varphi + \pi, r \to r$; $\hat{s}_2 : \theta \to \theta + \pi, \varphi \to \varphi, r \to -r$; $\hat{s}_3 : \theta \to -\theta + \pi, \varphi \to \varphi + \pi, r \to -r$ (the full symmetry group of (4.9) is obtained by adding to $\hat{s}_a$ the transformation $\theta \to \theta + 2\pi n$, $\varphi \to \varphi + 2\pi m$, where $m, n \in \mathbb{Z}$). Thus, two different transformations of ξ correspond to every $\mathbb{Z}_2$-transformation $r \to \pm r$, i.e. $S^* \sim \mathbb{Z}_2 \otimes \mathbb{Z}_2$ in $\mathscr{H}_{\mathrm{ph}}$. Moreover, $\hat{T}_s = \hat{s}$ since both the representations of bosons and of fermions coincide.

All physical states must be invariant under S^* because S^* is a subgroup of the gauge group SO(3). We may also prove it in another way. Consider an eigenfunction φ_E of the full Hamiltonian (without constraints) $\frac{1}{2}p_a^2 + V(x, \psi^+, \psi)$ in the coordinates (4.9). In accordance with (2.10) φ_E should be invariant under the above-described discrete symmetry group for the change of variables (4.9). The dependence of φ_E on θ and φ is determined by the spherical functions $Y_{lm}(\theta, \varphi)$ because $V(x, \psi^+, \psi) = V(\rho, \xi^+, \xi)$ (V is gauge invariant), i.e. $\varphi_E = \sum R^E_{lm} Y_{lm}$. So, R^E_{00} are invariant under S^* but they form a basis in $\mathscr{H}_{\mathrm{ph}}$, thus any Φ is invariant under S^* if $\Phi \in \mathscr{H}_{\mathrm{ph}}$. Note, (4.12) is fulfilled automatically for R^E_{00}. Thus, we write the unit operator kernel in $\mathscr{H}_{\mathrm{ph}}$ by analogy with (2.12a):

$$\langle q|q'\rangle_{\mathrm{ph}} = \frac{1}{2rr'}[\delta(r-r')(\mathrm{e}^{\bar{\xi}\xi'} + \mathrm{e}^{\bar{\xi}\hat{s}_1\xi'}) - \delta(r+r')(\mathrm{e}^{\bar{\xi}\hat{s}_2\xi'} + \mathrm{e}^{\bar{\xi}\hat{s}_3\xi'})]. \tag{4.13a}$$

The factor $\frac{1}{2}$ in (4.13a) follows from the equality

$$\int_0^\infty \mathrm{d}r\, r^2 \int \mathrm{d}\bar{\xi}\, \mathrm{d}\xi\, \mathrm{e}^{-\bar{\xi}\xi} \langle q'|q\rangle_{\mathrm{ph}} \Phi(q) \equiv \Phi(q') \tag{4.13b}$$

where $\Phi \in \mathscr{H}_{\mathrm{ph}}$, $r \in \mathbb{R}$ and $r' > 0$ in (4.13a). Of course, (4.13a) can be obtained from (2.12) by averaging over θ and φ since θ and φ are non-physical variables.

The PI derivation for U_t^{ph} coincides with (2.13)–(2.31) if we replace H by H_{ph} (see (4.11)) and $\langle q|q'\rangle$ by (4.13a). A final expression has the form (2.31) where $\hat{Q}$ is given by the expression in the brackets of (4.13a) with factor $\frac{1}{2}$ if the sign of $\delta(r+r')$ is changed, and

$$H^{\mathrm{eff}} = \tfrac{1}{2}p^2 + \tilde{V}(r, \bar{\xi}, \xi) - \frac{1}{2r^2}[(\bar{\xi}\varepsilon_2\xi)^2 + (\bar{\xi}\varepsilon_3\xi)^2 + \bar{\xi}(\varepsilon_2^2 + \varepsilon_3^2)\xi]. \tag{4.14}$$

Here $\tilde{V}$ is defined as in (2.19), matrix elements of the matrix ε_a are ε_{abc}, and p is a momentum canonically conjugated to r.

The main point we would like to make is that the PI contains the operator $\hat{Q}$ symmetrizing the transition amplitude over the group S^*. It was shown (Prokhorov and Shabanov 1989, Shabanov 1989a, 1991) that $\hat{Q}$ appears for gauge systems when a physical phase space reduction takes place. On the other hand, by construction the kernel $U_t^{\mathrm{ph}}(q, \bar{q}')$ ($q = (r, \bar{\xi})$) is invariant under S^*. Then, we state that q and $\bar{q}'$ in it can be replaced by Q and $\bar{Q}'$ respectively ($Q = (x, \bar{\psi})$), and the result does not depend on the non-physical variables θ and φ, i.e.

$$U_t^{\mathrm{ph}}(q, \bar{q}') = U_t^{\mathrm{ph}}(Q, \bar{Q}'). \tag{4.15}$$

In other words, there exists one-to-one gauge-invariant analytical continuation of the kernel U_t^{ph} to the total configuration space of the system. To prove this, we note that

any polynomial of q invariant under S^* depends only on degrees of the following quantities

$$\delta_{ab}\rho_a\rho_b, \qquad \varepsilon_{abc}\rho_a\bar{\xi}_b\bar{\xi}_c \qquad \delta_{ab}\rho_a\bar{\xi}_c \qquad \varepsilon_{abc}\bar{\xi}_a\bar{\xi}_b\bar{\xi}_c \tag{4.16}$$

where $\rho = (r, 0, 0)$. We may check this directly. Since δ_{ab} and ε_{abc} are invariant tensors of SO(3) (Barut and Raczka 1977), we conclude that quantities (4.16) are equal respectively to

$$x_a^2 \qquad \varepsilon_{abc}x_a\bar{\psi}_b\bar{\psi}_c \qquad x_a\bar{\psi}_a \qquad \varepsilon_{abc}\bar{\psi}_a\bar{\psi}_b\bar{\psi}_c \tag{4.17}$$

in accordance with (4.9). Any gauge-invariant polynomial can be formed from (4.17) (compare with (4.8)!). Moreover, an analytical function of q being invariant under the residual discrete gauge group S^* has the unique analytical gauge-invariant continuation to the space of Q because polynomials form a dense set in the space of analytical functions. So, (4.15) is proved. Note, Q contains six degrees of freedom and a gauge arbitrariness has three parameters; nevertheless, the system has four physical degrees of freedom (see (4.8) or (4.17)). This happens because two first constraints in (4.10) already pick out the full $\mathcal{H}_{\text{ph}}$, as has been shown above.

Thus, the explicit gauge-invariant form of the PI for a transition amplitude can be obtained if we take into consideration a curvilinearity of physical variables and their phase space reduction. Both of these main moments are usually ignored in the standard PI derivation for gauge theories.

5. The case of an arbitrary group and generalized Shevalley theorem

Here we attempt to reveal a general mathematical origin of equality (4.15). It turns out that a statement like the Shevalley theorem (Partasarathy *et al* 1967) makes a basis of equality (4.15) in the general case.

Consider the model with the Lagrangian

$$L = \tfrac{1}{2}\,\mathrm{Tr}(D_t x)^2 + \mathrm{i}\,\mathrm{Tr}\,\psi^+ D_t\psi - V(x, \psi^+, \psi). \tag{5.1}$$

Here $D_t = \partial_t + [y, \,]$; variables x, y, ψ^+, ψ are elements of a Lie algebra X of an arbitrary compact gauge group G, i.e. $x = x_i\lambda_i$ (analogously for y), $\psi = \lambda_i\psi_i$ (analogously for ψ^+), x_i, y_i are real, ψ_i, ψ_i^+ are complex Grassman variables, where λ_i is an orthonormal basis in X: $\mathrm{Tr}\,\lambda_i\lambda_j = \delta_{ij}$, $[\lambda_i, \lambda_j] = f_{ijk}\lambda_k$, f_{ijk} are total antisymmetric structural constants and $i, j, k = 1, 2, \ldots, N = \dim X$. Lagrangian (5.1) is invariant under gauge transformations

$$\begin{aligned} &x \to \Omega x\Omega^{-1} \qquad \psi \to \Omega\psi\Omega^{-1} \qquad \psi^+ \to \Omega\psi^+\Omega^{-1} \\ &y \to \Omega y\Omega^{-1} + \Omega\partial_t\Omega^{-1} \end{aligned} \tag{5.2}$$

where $\Omega = \Omega(t) \in \mathrm{G}$, and we assume that V is invariant under (5.2).

Canonical momenta are $\pi = \partial L/\partial\dot{y} = 0$, $p = \partial L/\partial\dot{x} = D_t x$. We describe Grassman degrees of freedom as in section 4, i.e. we introduce the Dirac brackets (4.4). So, the Hamiltonian is

$$H = \tfrac{1}{2}\,\mathrm{Tr}\,p^2 + V(x, \psi^+, \psi) + y_i\mathrm{G}_i \tag{5.3}$$

where $\mathrm{G}_i = -\{\pi_i, H\} = f_{ijk}(p_j x_k + \mathrm{i}\psi_j^+\psi_k) = 0$ are the secondary constraints. As one may check, they are the first-class constraints. After a quantization of the theory, G_i choose the physical subspace $\mathcal{H}_{\text{ph}}$:

$$\mathrm{G}_i|\Phi_{\text{ph}}\rangle = \pi_i|\Phi_{\text{ph}}\rangle = 0. \tag{5.4}$$

Our purpose is a PI construction for the evolution operator kernel of physical degrees of freedom. In accordance with the method suggested in section 4 it is necessary to introduce new curvilinear coordinates in which the constraints (5.4) are diagonalized, then to write the Hamiltonian in $\mathscr{H}_{\rm ph}$ and to find $\langle q|q'\rangle_{\rm ph}$. Finally, $U_t^{\rm ph}(q,\bar{q}')$ may be restored by the method of section 2.

Determine new variables as follows (Prokhorov and Shabanov 1989)

$$x = {\rm e}^{z} h\, {\rm e}^{-z} \qquad \psi = {\rm e}^{z}\xi\, {\rm e}^{-z} \tag{5.5}$$

where $h \in H$ is a Cartan subalgebra in X (Helgason 1984) $z \in X \ominus H$. In accordance with (5.2) z are non-physical variables. Note that like (4.9) h has a stationary subgroup in G, the Cartan subgroup, i.e. maximal Abelian subgroup in G (Helgason 1984). We denote $h = h_\alpha \lambda_\alpha$ ($\alpha = 1, 2, \ldots, l = \dim H$), $z = z_a \lambda_a$ ($a = l+1, l+2, \ldots, N$). The metric in the new variables has the block-diagonal form (Prokhorov and Shabanov 1989, Shabanov 1989a) $g^{ij} = (\delta_{\alpha\beta}, [(F^T\omega^T\omega F)^{-1}]_{a\dot{b}})$ where $\omega_{a\dot{b}} = h_\alpha f_{\alpha a\dot{b}}$, $F_{i\dot{a}} = {\rm Tr}(\lambda_i\, {\rm e}^{z}\partial_a\, {\rm e}^{-z})$ and $\partial_a = \partial/\partial z_a$. The measure is ${\rm d}x = \det \omega F\, {\rm d}h\, {\rm d}z \equiv \mu^2(h)\tilde{\mu}(z)\, {\rm d}h\, {\rm d}z$. The measure in a physical configurational space may be calculated explicitly (Helgason 1984):

$$\mu(h) = \prod_{\alpha>0} (h, \alpha) = (\det \omega)^{1/2} \tag{5.6}$$

where α are positive roots of X, $(h, \alpha) = h_\beta \alpha_\beta$.

To find the Hamiltonian in $\mathscr{H}_{\rm ph}$, we calculate the constraints in new variables. Since z_a are translated under gauge transformations generated by constraints, $N-l$ constraints G_j are linear combinations of $i\partial_a$ (compare with (4.10)). The remaining gauge arbitrariness is connected with the Abelian one-dimensional Cartan group which *does not change the physical boson variables* h_α, *but changes the fermion variables* ξ. So, other l constraints must represent the equalities to zero of generators of Abelian transformations of fermion variables (like (4.10)). Thus, equations (5.4) are equivalent to

$$-{\rm i}\partial_a \Phi_{\rm ph} = 0 \qquad L_\alpha \Phi_{\rm ph} \equiv f_{\alpha a\dot{b}} \xi_a^+ \xi_{\dot{b}} \Phi_{\rm ph} = 0 \tag{5.7}$$

where $\xi_{\dot{b}} = \partial/\partial\bar{\xi}_{\dot{b}}$. Note that $f_{\alpha\beta i} = 0$, hence $[L_\alpha, L_\beta] = 0$, i.e. L_α are generators of the Cartan subgroup.

In the quantum Hamiltonian (5.3) rewritten in the form (2.6) for coordinates (5.5) we carry ∂_a and L_α to the right and use (5.7) in $\mathscr{H}_{\rm ph}$; we then get the quantum Hamiltonian in $\mathscr{H}_{\rm ph}$. To simplify calculations, note that in new variables diagonalizing constraints, non-physical variables become cyclic (Dirac 1965), i.e. $H_{\rm ph}$ does not depend on them. So, we may only keep an eye on terms containing h and ξ^+, ξ. We have

$$H_{\rm ph} = -\frac{\rm i}{2\mu}\partial_\alpha \circ \mu\partial_\alpha + \tfrac{1}{2} L_a (\omega^T\omega)^{-1}_{a\dot{b}} L_{\dot{b}} + V(h, \xi^+, \xi). \tag{5.8}$$

Here $\partial_\alpha = \partial/\partial h_\alpha$, $L_a = -{\rm i} f_{aij}\xi_i^+\xi_j$.

To find S and S^*, we introduce the Cartan–Weyl basis in X (Barut and Raczka 1977):

$$[e_\alpha, e_{-\alpha}] = \alpha \qquad [h, e_\alpha] = (\alpha, h) e_\alpha \qquad [e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta} \tag{5.9}$$

where $\alpha > 0$ are positive roots in X, e_α are corresponding root vectors, h, $\alpha \in H$, $N_{\alpha\beta}$ are numbers, $N_{\alpha\beta} \neq 0$ if $\alpha+\beta$ is a root in X. We also define an operator of the adjoint representation ad $x(y) = [x, y]$ for all x, $y \in X$. Any element $h \in H$ can be represented as $h = h_\omega \omega$ where ω are simple roots of X, hence, the set $\{\omega, e_\alpha, e_{-\alpha}\}$ gives a basis in

X (Cartan–Weyl basis). However, it is more convenient for us to use the orthogonal basis in $X \ominus H$:

$$s_\alpha = \frac{1}{\sqrt{2}}(e_\alpha - e_{-\alpha}) \qquad c_\alpha = \frac{1}{\sqrt{2}}(e_\alpha + e_{-\alpha}). \tag{5.10}$$

(The orthogonality is understood with respect to the scalar product in X: $(x, y) = \mathrm{Tr}\ \mathrm{ad}\ x\ \mathrm{ad}\ y$; for compact groups one may normalize a basis in X so that $(x, y) = \mathrm{Tr}\ xy$ in a matrix representation (Barut and Raczka 1977).)

It is well known that there exists a subgroup of G in H called the Weyl group W which is a group of reflections and rearrangements in the root system. The group W is defined by combinations of the operators (Zhelobenko 1970)

$$\hat{s}^s_\omega = \exp \frac{\pi}{(\omega, \omega)^{1/2}} \mathrm{ad}\ s_\omega \qquad \hat{s}^c_\omega = \exp \frac{\mathrm{i}\pi}{(\omega, \omega)^{1/2}} \mathrm{ad}\ c_\omega \tag{5.11}$$

i.e. any $\hat{s} \in \mathrm{W}$ is a combination of $\hat{s}^s_\omega$ or a combination of $\hat{s}^c_\omega$ (ω are simple roots). We may check that $\hat{s}^{c,s}_\omega \omega = -\omega$, i.e. (5.11) are reflections of all simple roots, and they give two equivalent representations of W in H. In accordance with the definition of ad x and (5.5) we conclude that actions of W in H induce transformations in $X \ominus H$, but the left-hand sides of (5.5) are invariant. Hence, transformations (5.11) are generators of a searched discrete group S. Indeed, the change of boson variables (5.5) exists if $h \in K^+ = H \backslash \mathrm{W}$ (Helgason 1984) where K^+ is the Weyl camera (physical configurational space (Prokhorov and Shabanov 1989)). In other words, S cannot contain generators except (5.11), otherwise $H \backslash S \subset K^+$, which is wrong. Note, $\hat{s}^c_\omega$ and $\hat{s}^s_\omega$ coincide in H but their actions are different for Grassman elements ξ.

We call the discrete group defined by (5.11) in space $H_g = X_g \oplus H (\bar{\xi} \in X_g, h \in H)$ the generalized Weyl group W*. Since boson and fermion representations are identical, $S^* = \mathrm{W}^*$. Certainly, to get a full symmetry group of the change of variables we must add to W* transformations of z inducing shifts $\mathrm{e}^z = \mathrm{e}^{z+a}$ like $2\pi n$-shifts of θ, φ in (4.9). Using considerations like above-suggested ones for the derivation of (4.13) (we denote N_* the number of different elements of S^* such that $\hat{s}^* q = (h, \bar{\xi}\hat{T}^+_s)$, $N_* = 2$ in (4.13a)) one may write

$$\langle q | q' \rangle_{\mathrm{ph}} = \frac{N_*^{-1}}{\mu(h)\mu(h')} \sum_{\mathrm{W}^* = S^*} (-1)^{p_s} \delta(q, \hat{s}^* \bar{q}') \tag{5.12}$$

where $q \in H_g$, $q' \in X_g \oplus K^+$ and $\mu(\hat{s}h) = (-1)^{p_s}\mu(h)$, $\hat{s} \in \mathrm{W}$, $p_s = 0$ if $\hat{s}$ is rearrangement of roots without reflections, $p_s = 1$ for $\hat{s}$ including non-even numbers of reflections of roots. Equality (5.12) means that all physical states from $\mathscr{H}_{\mathrm{ph}}$ are invariant under the residual discrete gauge group W*. Moreover, the requirement of the W*-invariance gives automatically solutions of constraints (5.7) in the Grassman sector. To prove this last statement, note that $\hat{s}^c_\omega \hat{s}^s_\omega = 1$ in H; however, in X these operators must be elements $\exp \mathrm{ad}\ \lambda$, $\lambda \in H$ which are equal to 1 in H. On the other hand, one may check by direct calculations on the basis of (5.9) that operators (5.11) are reflections with rearrangements in the real basis of $X \ominus H$ ($\mathrm{i}c_\alpha$, s_α), $\alpha > 0$ (Zhelobenko 1970). Then $\exp \mathrm{ad}\ \lambda$ are also combinations of rearrangements and reflections. Using this we can find explicit forms of λ. Indeed, $\exp(\mathrm{ad}\ \lambda)\mathrm{i}s_\alpha$ is only $\pm \mathrm{i}s_\alpha$ or $\pm c_\alpha$, as follows from (5.9) and $\lambda \in H$. So, λ can take values $\mathrm{i}\pi\alpha(\alpha, \alpha)^{-1}$, α runs over all positive roots, i.e. W* contains the operators $\hat{s}_\alpha = \exp \mathrm{i}\pi(\alpha, \alpha)^{-1}\ \mathrm{ad}\ \alpha$. Further, transformations from the Cartan subgroup $\exp \mathrm{ad}\ \chi$ ($\chi \in H$) generated by L_α in (5.7) on the basis of (5.10) ($\bar{\xi} = \bar{\xi}_\omega \omega + \bar{\xi}^c_\alpha c_\alpha + \bar{\xi}^s_\alpha s_\alpha$) are rotations of two-dimensional Grassman vectors ($\bar{\xi}^c_\alpha$, $\bar{\xi}^s_\alpha$) through the angle (χ, α)

for every $\alpha > 0$. Invariants of these rotations are $\bar{\xi}^c_\alpha \bar{\xi}^s_\alpha$ (α is fixed), but $\hat{s}_\alpha c_\alpha = -c_\alpha$, $\hat{s}_\alpha s_\alpha = -s_\alpha$, hence $\bar{\xi}^c_\alpha \bar{\xi}^s_\alpha$ are also invariant under W*, i.e. W*-invariant functions give solutions of (5.7) in the Grassman sector.

Using the technique of section 2, we restore the form of $U^{\rm ph}_t(q, \bar{q}')$ for Hamiltonian (5.8) and kernel (5.12). It has the form (2.31) where

$$Q(q, \bar{q}') = \sum_{W^*=S^*} N_*^{-1} \delta(q, \hat{s}^* \bar{q}') \tag{5.13}$$

$$H^{\rm eff} = \tfrac{1}{2} p_\alpha^2 + \tilde{V}(h, \bar{\xi}, \xi) + \tfrac{1}{2} L_a (\omega^T \omega)^{-1}_{ab} L_b + V_q \tag{5.14}$$

and $L_a = -{\rm i} f_{aij} \bar{\xi}_i \xi_j$, $V_q = -\bar{\xi}_i \xi_n f_{aik} f_{bnk} (\omega^T \omega)^{-1}_{ab}$. The constructed kernel $U^{\rm ph}_t$ turns out to be invariant under W* like (4.15) ($\hat{Q}$ symmetrizes it in W*). If fermions are absent, W* = W. In this case, from the Shevalley theorem (Zhlobenko 1970), every analytical function in H being invariant under W has the unique analytical gauge-invariant continuation to X. So, $U^{\rm ph}_t(h, h') = U^{\rm ph}_t(x, x')$. Examples of the construction of gauge-invariant wavefunctions were given in Shabanov (1989c) and gauge-invariant forms of PIs in total (i.e. including also non-physical degrees of freedom) configurational and phase spaces were presented in Shabanov (1989c, 1991).

For the present system there is an analogous statement, which we call the generalized Shevalley theorem: every analytical function in $H_{\rm g}$ being invariant under W* has the unique analytical gauge-invariant continuation to $X \oplus X_{\rm g}$ ($Q \in X \oplus X_{\rm g}$ if $x \in X$, $\bar{\psi} \in X_{\rm g}$). Consider an oscillator in (5.3), $V(h, \xi^+, \xi) = \frac{1}{2} {\rm Tr}\, h^2 + {\rm Tr}\, \xi^+ \xi - N/2$. Its wavefunctions are $p_{\rm E}(q) \exp(-\frac{1}{2} {\rm Tr}\, h^2)$ where $p_{\rm E}(q)$ are polynomials invariant under W*. Since $H_{\rm ph}$ is Hermitian, $p_{\rm E}(q)$ form a basis in the space of all W*-invariant polynomials in $H_{\rm g}$. On the other hand, we may solve the quantum problem in the total Hilbert space, i.e. in the space of functions in $X \oplus X_{\rm g}$. Then, eigenfunctions of the oscillator are $\tilde{P}_{\rm E}(Q) \exp(-\frac{1}{2} {\rm Tr}\, x^2)$, moreover, $\mathscr{H}_{\rm ph}$ is formed by gauge-invariant polynomials from $\tilde{P}_{\rm E}(Q)$ which give a basis in the space of all gauge-invariant polynomials (the total Hamiltonian is also Hermitian). Because V is gauge invariant, we may write in coordinates (5.5) $\tilde{P}_{\rm E}(Q) = \Sigma_n P^n_{\rm E}(q) Y_n(z)$ where $Y_n(z)$ are eigenfunctions of the Laplace–Beltrami operator on a gauge group orbit formed by values of z when h is fixed. Clearly, $p_{\rm E}(q) = P^0_{\rm E}(q)$ ($Y_0 =$ constant). Then, in $\mathscr{H}_{\rm ph} \tilde{P}_{\rm E}(Q(q)) = \tilde{P}_{\rm E}(q) = P^0_{\rm E}(q) = p_{\rm E}(q)$ because of the gauge invariance, i.e. between polynomials $P^0_{\rm E}$ and $p_{\rm E}$ there exists a one-to-one correspondence, hence it exists between $\tilde{P}_{\rm E}(Q) \in \mathscr{H}_{\rm ph}$ and $p_{\rm E}(q)$. Since polynomials form a dense set in the space of analytical functions, we arrive at the statement of the generalized Shevalley theorem. Thus, (4.15) takes place in the general case.

Note a simple consequence. Every polynomial in $X_{\rm g}$ being invariant under W* is gauge invariant, i.e. a gauge symmetry in a pure fermion sector of a theory is equivalent to the discrete symmetry with respect to the generalized Weyl group W*.

6. Conclusion

We have seen that the main points of PI derivation corresponding uniquely to the Dirac quantization scheme (i.e. to an explicit gauge-invariant description) are the curvilinearity of physical variables and reduction of both physical configuration and phase spaces. The latter, as has been shown, is connected with the invariance of PIs under residual discrete gauge transformations (the operator $\hat{Q}$ in the expression of

$U_t^{\rm ph}$), and this guarantees an explicit gauge invariance of PIs (the generalized Shevalley theorem).

The method may be generalized to any theory with first-class constraints (i.e. to any gauge theory). Let independent constraints be φ_a which generate gauge transformations (Pyatov and Razumov 1989). The structure of gauge group orbits in the total configurational space is not always known, therefore physical variables are picked out with the help of supplementary conditions $\chi_a(x)=0$. To get correspondence to the Dirac scheme, one has to do as follows. Let the gauge transformation law be $x \to ux$, $\psi \to T_u\psi$ where $u \in G$, G is a gauge group, T_u is a representation of G. Then, after quantization we change variables in a quantum Hamiltonian $(x, \psi) \to (\theta, y, \xi)$ where $x = u(\theta)y$, $\psi = T_u(\theta)\xi$, and y satisfies supplementary conditions $\chi_a(y)=0$. In this case constraints φ_a become linear combinations of derivatives $\partial/\partial\theta_a$ since θ_a shift under gauge transformations, i.e. θ_a are non-physical variables. Further, one should define a quantum Hamiltonian in the physical subspace, i.e. in the space of analytical functions of y, and find a unit operator kernel in the physical subspace of states, i.e. determine the measure (Jacobian) and the group S* (the group S may be found from conditions $\chi_a(\hat{s}y)=0$, $\hat{s} \in G$ where $\hat{s}$ are all residual discrete gauge transformations keeping conditions $\chi_a=0$). Finally, $U_t^{\rm ph}$ can be restored in accordance with the above-suggested method. The effective-action form and S* depend on the χ_a form. However, changing χ_a by χ'_a is equivalent to a passage to other curvilinear coordinates in quantum theory unbreaking, however, the diagonality of quantum constraints ($x = uy = u'y'$, $\chi'_a(y')=0$ and $\varphi_a \sim \partial/\partial\theta_a \sim \partial/\partial\theta'_a$), hence it is a passage to a new basis in $\mathscr{H}_{\rm ph}$. So, the change of χ_a does not influence the form of the function $U_t^{\rm ph}$ which depends only on gauge-invariant quantities (see (4.15)). Change of χ_a is the change in form of an entry of gauge-invariant quantities (compare (4.16) with (4.17), in this case $\chi_a=0$ are $x_2 = x_3 = 0$).

It is necessary to say, that a quantum theory determined by the elimination of non-physical variables with subsequent quantization and the one found in accordance with the Dirac scheme are free from internal contradictions, nevertheless they can be different. Therefore we may consider them as two quantum versions of the same classical theory. However, note that in the case of a quantum gauge field theory we should observe an explicit Lorentz invariance in choosing physical variables. The latter is known to require the introduction of non-physical variables to a theory (Dirac 1967). Otherwise, we cannot impose supplementary conditions on operators since contradictions with commuting relations appear (Dirac 1965, 1967). Therefore the Dirac scheme turns out to be more preferable for formulation of a theory in the total Hilbert space as it is free from these contradictions. Thus, PIs should be defined according to the Dirac quantization scheme.

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